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The flat plate boundary layer. Part 2. The effect of increasing thickness on stability

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Numerical analysis has been used to find the neutral stability curve for the flat plate boundary layer in zero pressure gradient when the main terms representing the growth of boundary-layer thickness are either included or excluded. The boundary layer is found to be slightly less stable when the extra terms are included. The calculations give a critical Reynolds number of 500.

Introduction

The neutral stability curve for the two-dimensional laminar boundary layer on a flat plate under zero pressure gradient has been calculated on large computers by Kurtz (1961), Kaplan (1964), Osborne (1967), Wazzan *et al.* (1968) and Jordinson (1970, hereafter referred to as part 1). The results of these calculations, obtained by slightly different methods, are sufficiently consistent to justify the view that the neutral-curve eigenvalues of the Orr–Sommerfeld equation for this flow are now well established. When these results are compared with the experimental observations of Schubauer & Skramstad (1947) and Ross *et al.* (1970) it is found that theory and experiment are in close agreement for boundary-layer Reynolds numbers of 1000 and over, but do not agree so well at lower Reynolds numbers. Possible reasons for a lack of agreement as the Reynolds number decreases may be suggested; on the one hand the experimental difficulties become greater and increasing experimental errors are unavoidable, and on the other hand in the theoretical analysis the 'parallel mean flow' approximation made in deriving the Orr–Sommerfeld equation tends to become less accurate. An estimate of the effect of the parallel flow assumption may be made by performing a numerical analysis of a modified form of the Orr–Sommerfeld equation in which the more important terms representing the growth of boundary-layer thickness are included. The present paper reports such a calculation.

The modified equations

In order to assess the relative magnitude of all the terms occurring in the complete equations, it is convenient to begin with the Prandtl equations for steady two-dimensional flow in the zero-pressure-gradient boundary layer. Using U_0 as the free-stream velocity (in the x direction), U as the x component of steady velocity in the boundary layer, W as the z component of steady velocity (normal to the plate), and P as the steady pressure, the relative magnitude of the Navier–Stokes terms may be found by substituting the non-dimensionalizing relations

$$\left.\begin{array}{lll} x = Lx', & U = U_0 U', & R_x = LU_0/\nu = xU_0/\nu, \\ z = LR_x^{-\frac{1}{2}} z', & W = U_0 R_x^{-\frac{1}{2}} W', & \\ t = LU_0^{-1} t', & P = \rho U_0^2 P', & \end{array}\right\} \tag{1}$$

the origin of x being taken at the leading edge of the plate. The equations then reduce to the Prandtl form

$$\left.\begin{array}{l} \dfrac{\partial U'}{\partial x'} + \dfrac{\partial W'}{\partial z'} = 0, \quad \dfrac{\partial P'}{\partial z'} = 0, \\ U' \dfrac{\partial U'}{\partial x'} + W' \dfrac{\partial U'}{\partial z'} = \dfrac{\partial^2 U'}{\partial z'^2}. \end{array}\right\} \tag{2}$$

All the Navier–Stokes terms which are retained are of first order, and all omitted terms are of order R_x^{-1} (or less) relative to the surviving terms in the equation concerned. This order of approximation is fully satisfactory since the experimental lower limit of R_x is about 40,000.

Following the method of Jones & Watson (1961), the Prandtl equations are reduced to the Blasius equation

$$f'''(\eta) + f(\eta) f''(\eta) = 0, \tag{3}$$

using the non-dimensional variable $\eta = (U_0/2\nu x)^{\frac{1}{2}} z$ and the dimensional stream function $\Psi = (2U_0 \nu x)^{\frac{1}{2}} f(\eta)$. The numerical integration of (3) with appropriate boundary conditions leads to the following results for the displacement thickness of the boundary layer, δ_1, the boundary-layer Reynolds number, R, and other variables required here.

$$\begin{aligned} &\delta_1 = 2^{\frac{1}{2}} x R_x^{-\frac{1}{2}} \lim_{\eta \to \infty} [\eta - f(\eta)] = m x R_x^{-\frac{1}{2}}, \quad \text{where} \quad m = 1{\cdot}7208, \\ &R = U_0 \delta_1/\nu = m R_x^{\frac{1}{2}}, \\ &U = U_0 f'(\eta) \quad \text{and} \quad \lim_{\eta \to \infty} f'(\eta) \to 1, \\ &W = (2R_x)^{-\frac{1}{2}} U_0 [\eta f'(\eta) - f(\eta)] \quad \text{and} \quad \lim_{\eta \to \infty} W = W_\infty \to \tfrac{1}{2} m R_x^{-\frac{1}{2}} U_0. \end{aligned}$$

When R_x is reduced to 40,000, W_∞ rises to $0{\cdot}0043\, U_0$.

When a two-dimensional perturbation is superposed on the Blasius flow, the equations for the combined flow may be expressed using u and w for the x and z components of the perturbation velocity, and H and h for the vorticities in the mean flow and the perturbation, respectively. The equations governing the total flow are those for continuity and vorticity. Assuming that the perturbation velocities are sufficiently small to justify the neglect of the non-linear terms $u(\partial h/\partial x)$ and $w(\partial h/\partial z)$ and subtracting the mean flow terms, the linearized equations for the perturbation are obtained.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

$$\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + u \frac{\partial H}{\partial x} + W \frac{\partial h}{\partial z} + w \frac{\partial H}{\partial z} = \nu \nabla^2 h. \tag{5}$$

Equation (4) is satisfied by a stream function ψ, and the periodic form

$$\psi = \phi(z) \exp i(\alpha x - \beta t)$$

is adopted here, with β taken to be purely real, and $\alpha = \alpha_r + i\alpha_i$. Substituting ψ in (5) and using $D\phi$ for $d\phi/dz$, we obtain the relation

$$(i\alpha U - i\beta + WD)(D^2 - \alpha^2)\phi + \left(\frac{\partial^2 U}{\partial x \partial z} - \frac{\partial^2 W}{\partial x^2}\right). D\phi$$
$$- i\alpha\left(\frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 W}{\partial x \partial z}\right)\phi = \nu(D^2 - \alpha^2)^2\phi. \quad (6)$$

In order to reduce (6) to the Orr–Sommerfeld form, the assumption $W = 0$ is required, with the consequent relation $\partial U/\partial x = 0$, implying parallel mean flow in the boundary layer. If this assumption is not made, the relative magnitude of all the terms in (6) may be found by substituting (1) together with $\alpha = L^{-1}R_x^{\frac{1}{2}}\alpha'$ and $\beta = L^{-1}R_x^{\frac{1}{2}}U_0\beta'$ (the new relations being required to retain the main Orr–Sommerfeld terms). Equation (6) then becomes

$$(i\alpha' U' - i\beta' + R_x^{-\frac{1}{2}}W'D')(D'^2 - \alpha'^2)\phi' + R_x^{-\frac{1}{2}}\left(\frac{\partial^2 U'}{\partial x' \partial z'} - R_x^{-1}\frac{\partial^2 W'}{\partial x'^2}\right)D'\phi'$$
$$- i\alpha'\left(\frac{\partial^2 U'}{\partial z'^2} - R_x^{-1}\frac{\partial^2 W'}{\partial x' \partial z'}\right)\phi' = R_x^{-\frac{1}{2}}(D'^2 - \alpha'^2)^2\phi'. \quad (7)$$

This shows that two terms in (6), $WD(D^2 - \alpha^2)\phi$ and $(\partial^2 U/\partial x \partial z) D\phi$, are of the same order as the viscous term on the right-hand side, and are the principal terms representing the growth of boundary-layer thickness.

For numerical analysis, the resulting equation is made non-dimensional in the normal manner for the Orr–Sommerfeld equation, writing $z = \delta_1 z'$, $\alpha = \alpha'/\delta_1$, $\beta = U_0\beta'/\delta_1$, $U = U_0 U'$, $W = U_0 W'$ and $R = U_0\delta_1/\nu$. The equation to be integrated is then

$$(\alpha' U' - \beta')(D'^2 - \alpha'^2)\phi' - iW'D'(D'^2 - \alpha'^2)\phi'$$
$$+ i\frac{\partial^2 W'}{\partial z'^2}D'\phi' - \alpha'\frac{\partial^2 U'}{\partial z'^2}\phi' = -\frac{i}{R}(D'^2 - \alpha'^2)^2\phi', \quad (8)$$

where the coefficient $-\partial^2 U'/\partial x' \partial z'$ has been replaced by $\partial^2 W'/\partial z'^2$, using the continuity equation.

The boundary conditions on $\phi(z)$

As in part 1, the boundary conditions express the requirement that the perturbation velocities vanish at $z = 0$ and $z = \infty$. At $z = 0$, $\phi = D\phi = 0$. For large values of z, (8) takes the form:

$$(\alpha' - \beta')(D'^2 - \alpha'^2)\phi' - iW'_\infty D'(D'^2 - \alpha'^2)\phi' = -(i/R)(D'^2 - \alpha'^2)^2\phi',$$

and the solution fitting the outer boundary conditions is

$$\phi' = A\,e^{-\alpha' z'} + B\,e^{-\mu z'},$$

where A and B are arbitrary constants, and $\mu^2 - \mu R W'_\infty - \gamma^2 = 0$, with $\gamma^2 = \alpha'^2 + iR(\alpha' - \beta')$. Since $RW'_\infty \approx 1{\cdot}5$ and $|\gamma^2|$ is of the order of 100, $|\mu| \approx |\gamma|$

and $|e^{-\mu z'}| \leqslant |e^{-\alpha' z'}|$ for $z' > 0$. The required outer boundary condition may therefor be expressed in the form $\phi' \sim e^{-\alpha' z'}$ for large z'. The presence of the W' term does not affect the form of the boundary conditions, and the calculations have been carried out on this assumption.

The eigenvalue calculations

The purpose of the analysis is to determine the eigenvalues $\alpha = \alpha_r + i\alpha_i$ for given R and given real β. The mean flow coefficients should be known as accurately as possible, and direct integration of (3) is therefore desirable. The computer program written by Jordinson for part 1 of this paper was kindly made available for the present work, and was supplemented by a program for the calculation of the new terms in (8). The resulting program was designed to give results both with and without the new terms so that an accurate comparison could be made.

The rational difference approximation, given by Osborne (1967) and employed by Jordinson, was used, yielding a heptadiagonal antisymmetric matrix of finite-difference coefficients. In order to carry out the iteration for an eigenvalue, a good initial approximation was necessary; it was assumed that the same initial approximation would be satisfactory for the calculations with and without the additional terms. This assumption worked well and gave rise to no difficulties.

In Jordinson's work various step lengths, h, were used, and he finally chose 80 steps for the full range of the calculations, $0 \leqslant z/\delta_1 \leqslant 6$. In the present case, 80, 100 and 120 steps were used in preliminary runs, and it was found again that 80 steps in the same range of z gave adequate accuracy.

The results of the calculations

The inclusion of the W terms produces a reduction in the values of α_r' which is almost independent of β' and varies significantly with R. The reduction amounts to about 0·003 for $R = 400$, about 0·002 for $R = 1000$ and about 0·0007 for $R = 2000$. From the experimental point of view, interest is centred on the variations of the wavelength, λ, with R, for a given frequency parameter, $F = \beta'/R$, and a given value of U_0. The wavelength is then proportional to the reciprocal of $\alpha_r = \alpha_r' R^{-1} U_0 \nu^{-1}$, where $U_0 \nu^{-1}$ is constant. Table 1 shows some of the results grouped to show the variation of α_r as a function of R when U_0 and F are constant.

The inclusion of the W terms affects the values of α_i' in a more complicated manner. In general, the values are reduced, and in the neighbourhood of branch I of the neutral stability curve (where α_i' first passes through zero) the reduction is roughly inversely proportional to R. At constant R the reduction in α_i' increases as β' increases, and this continues for some distance at least into the second damping region, but in the first damping region as β' decreases below its branch I value, the reduction in α_i' falls to a minimum and rises again. Since a negative value of α_i' gives amplification, a reduction of α_i' represents increased amplification or decreased damping. The new neutral stability curve therefore lies outside the curve obtained by Jordinson, and the critical Reynolds number is lower than his,

$F \times 10^6$	R	β'	$\alpha'_r R^{-1} \times 10^4$ — W terms included	$\alpha'_r R^{-1} \times 10^4$ — W terms excluded
40	500	0·02	1·36	1·41
	1000	0·04	1·32	1·32
	1500	0·06	1·27	1·27
	2000	0·08	1·26	1·26
50	400	0·02	1·60	1·70
	800	0·04	1·61	1·63
	2000	0·10	1·51	1·52
80	500	0·04	2·46	2·51
	1000	0·08	2·30	2·32
	1250	0·10	2·29	2·30
	1500	0·12	2·27	2·28
100	400	0·04	3·00	3·08
	800	0·08	2·81	2·83
	1000	0·10	2·79	2·80

TABLE 1. Variation of $\alpha'_r R^{-1}$ with R at constant F.
$F = \beta'/R = \beta\nu/U_0^2$. $\alpha'_r R^{-1} = \alpha\nu/U_0$

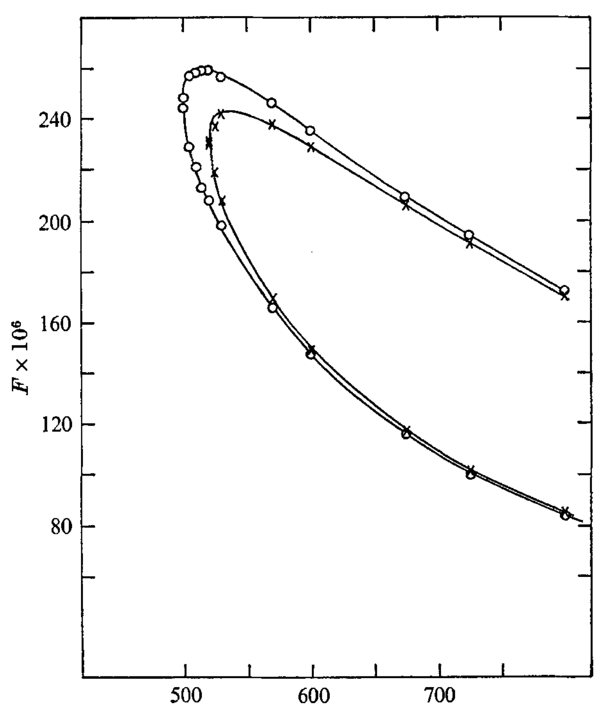

FIGURE 1. The calculated neutral stability curve: $F \times 10^6$ *versus* R, O, including the W terms; ×, excluding the W terms.

with a new value of just below 500. In order to define the neutral curve near its turning point, the program was run at closely spaced values of β' with the following values of R: 490, 500, 505, 510, 515, 520 and 530. The resulting data are given in table 2 and figure 1. The maximum value of F now obtained is about 260×10^{-6}.

R	$F \times 10^6$ Branch I	$F \times 10^6$ Branch II
500	244 (—)	248 (—)
505	229 (—)	257 (—)
510	221 (—)	258 (—)
515	213 (—)	259 (—)
520	208 (230)	259 (231)
530	198 (208)	256 (242)
570	166 (170)	246 (238)
600	147 (150)	235 (229)
675	116 (117)	209 (206)
725	100 (101)	194 (191)
800	84 (85)	172 (170)
1000	56 (56)	130 (129)
1250	38 (38)	98 (98)
1500	27 (27)	77 (76)
2000	17 (17)	52 (52)

TABLE 2. Points on the neutral stability curve calculated with W terms included; the numbers in brackets are calculated with equal accuracy with W terms excluded

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